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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, MARCH 2022
SECOND YEAR [BATCH 2020-23]

Date : 05/03/2022 **MATHEMATICS**
Time : 11am-1pm **Paper** : MTMA CC6 Full Marks : 50

Group A (3D Geometry)

Answer Question No. 1 and any two from Question No. 2 to Question No. 4.

1. Reduce the equation

$$3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$$

by matrix method to the canonical form and identify the surface it represents. [5]

2. Find the locus of the centres of the spheres which touch the straight lines [5]

$$y = mx, z = c \quad \text{and} \quad y = -mx, z = -c.$$

3. Find the condition that the straight lines in which the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ are perpendicular. [5]

4. Find the points of contact of the tangent planes to the conicoid $2x^2 - 25y^2 + 2z^2 = 1$, which pass through the line joining the points $(-12, 1, 12)$ and $(13, -1, -13)$. [5]

Group B (Multi-variable Calculus 1)

Answer any three questions from Question No. 5 to Question No. 9.

5. Let $f(x,y,z) = \begin{cases} xyz\frac{x^2-y^2+z^2}{x^2+y^2+z^2}, & \text{if } x^2+y^2+z^2 \neq 0 \\ 0, & \text{otherwise} \end{cases}$. Evaluate the following limits, if they exist.

(a) $\lim\limits_{(x,y)\to(0,0)} f(x,y,z)$, [2]

(b) $\lim\limits_{x\to 0}\lim\limits_{y\to 0}\lim\limits_{z\to 0} f(x,y,z)$ and [1.5]

(c) $\lim\limits_{y\to 0}\lim\limits_{z\to 0}\lim\limits_{x\to 0} f(x,y,z)$. [1.5]

6. (a) Determine whether $f_{xy}(0,0) = f_{yx}(0,0)$ for

$$f(x,y) = \begin{cases} \sqrt{x^2+y^2}\sin 2\theta & , \text{ if } x \neq 0 \\ 0 & , \text{ otherwise,} \end{cases}$$

where $\tan\theta = \frac{y}{x}$. [2.5]

(b) Find the directional derivative of the scalar valued function

$$f(x,y) = \begin{cases} x\sin\left(\frac{x+y}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

at the origin, in the direction $\beta = (1,1)$. [2.5]

7. (a) Show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable on a domain D, then the directional derivative of f at $(a, b) \in D$ in the direction of $\beta \in \mathbb{R}^2$ (unit vector) is equal to the projection of $grad(f)$ in the direction of β. [2]

   (b) Can we relax the differentiability condition in the above result? Justify. (No marks without proper justification) [2]

   (c) Can we conclude that the differentiability condition is the necessary condition? [1]

8. Let $(a, b) \in \mathbb{R}^2$. For a fixed $\delta > 0$ the l^∞ and l^2 neighborhoods of (a, b) are defined respectively as

$$U_\delta = \{(x, y) \in \mathbb{R}^2 : max|x - a|, |y - b| < \delta\} \text{ and}$$
$$V_\delta = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < \delta\}$$

Show that for each fixed $\delta > 0$, there exists γ and λ, both positive real numbers, such that $U_\gamma \subseteq V_\delta \subseteq U_\lambda$. [5]

9. (a) If $f, g, h : \mathbb{R} \to \mathbb{R}^3$ are three vector valued functions of a scalar variable x, such that [2]

$$\frac{df}{dx} = h \times f \text{ and } \frac{dg}{dx} = h \times g$$

then show that

$$\frac{d}{dx}(f \times g) = h \times (f \times g)$$

   (b) Find the unit vector perpendicular to the surface $x^2 + y^2 - z^2 = 12$ at the point (2,3,1). [3]

Group C (ODE 2)

Answer any two from Question No. 10 to Question No. 12

10. (a) Solve the equation [5]

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - 8)y = x^2e^{-x^2/2}$$

   (b) Solve the equation [5]

$$\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2y} = \frac{dz}{2z(x^2 + y^2)}$$

11. (a) Find the eigen-values and the corresponding eigen-functions of the boundary value problem [6]

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + \lambda y = 0, \quad y(1) = 0, y(e) = 0.$$

   (b) Solve $(z + z^2)\cos x\, dx - (z + z^2)dy + (1 - z^2)(y - \sin x)dz = 0$. [4]

12. (a) Solve the simultaneous equations [5]

$$\frac{d^2x}{dt^2} + 4x + y = te^t$$
$$\frac{d^2y}{dt^2} + y - 2x = \sin^2 t$$

   (b) Find the series solution of the equation

$$\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2 + 2x + 2$$

near the point $x = 0$. [5]

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